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BRIEF SURVEY OF QUANTUM MODELS AND POSSIBLE NEGATIVE MASS SOLUTIONS

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INTRODUCTION

Matter in the Universe falls (grossly) into two categories: macroscopic bodies in the world of the large, and, microscopic bodies in the world of the small. (There are of course a wide variety of objects that fall in between, but they are not the subject of this survey.)

By the mid 1800s, the macroscopic approach to theoretically describing matter interaction was the well-accepted answer [1]. That answer took the form of Newton's laws of gravity. Newton's model assumes all interacting particles are point sources (small radius relative to separation distance – extended sources can of course be handled by finite element techniques, but that also is a different story).

Particles (big or small) are known to be largely coherent densities of mass and composite in nature (except for elementary particles like the electron which are considered fundamental). Their position is identified as the location of their maximum mass density in space associated with a potential energy. In Newton's theory of gravity [2],

(1) $$\boldsymbol{V} = -\mathrm{G}m/\boldsymbol{r}$$

where $\boldsymbol{V}$ is the potential energy, G is the gravitational constant, $\boldsymbol{r}$ is the distance to the measurement point, and m is the mass of the particle (**bold** characters indicate vectors). The density profile of the particle is:

(2) $$\nabla^2 \boldsymbol{V} = 4\pi\rho$$

Classical macroscopic theory describes the particle's energy in terms of its position in time. For example,

(3) $$E = \boldsymbol{p}^2/2m - \boldsymbol{V}.$$

where $\boldsymbol{p} = m\boldsymbol{v}$, the particle's momentum when $\boldsymbol{v}$ is its velocity (as vectors, both $\boldsymbol{p}$ and $\boldsymbol{V}$ are position and time dependent).

By the mid 1920s, two things had happened. Einstein had developed his Theory of Relativity [3-4]. His Special Theory addressed the affects of speed on a particle's energy, motion and mass (among other things) which impacted the description of both large and small particles.

His General Theory addressed the nature of space and time in a way the classical picture did not (this is not included in this survey, but is the subject of much research on developing a unifying picture of the small and the large worlds).

The second thing that happened was that experiments indicated that the world of the small was much stranger than the macroscopic world described by Newton. The very tiny particles, like the electron, were known to have a property called spin. The classical notion of the spinning top could not be used to explain the spin of these particles because, as point sources, they would require a spin speed greater than the speed of light – something relativity theory prohibited. The other weird thing recognized about these particles was that they had a dual nature [5]. Under some conditions they acted like coherent particles, and, under other conditions, they acted like waves – enter the need for quantum mechanics – a new way to describe the world of the very small.

In the following sections, we discuss various quantum mechanical models – the classical Schrödinger equation, as well as, developments into the relativistic domain, that is, the relativistic Schrödinger equation, the Klein-Gordon equation and the Dirac equation. The discussion will include the assumptions, the requirements, including some discussion of the rational for the extension from previous pictures.

Before proceeding, we make one last comment about the nature of mass itself. Traditionally, mass was something that could be seen and touched or at least measured (in the form of the effects of the force it exerted on other masses). It was, therefore, accepted, by assumption, to be positive. However, there is nothing in Newton's model prohibiting it from being negative. One main difficulty with negative mass is a lack of a generally accepted intuitive interpretation of the negative mass concept; though, it is generally assumed that negative masses behave like positive masses – equal but opposite in their interaction.

The concept of negative mass has been considered as early as the late 1800s [1]. More recently it has been studied in in the context of General Relativity [6-7], and cosmological models showing equal and opposite mass particle creation [8]. In addition, single-metric [9], double-metric [10], and heuristic [11] dual universe models show four sets of particles (rather than the two of the Standard Model) based on both positive and negative mass – positive mass particles and anti-particles, and negative mass particles and anti-particles. Further, experiments are beginning to address the negative-mass question [12-17]

We now continue to consider the possibility of negative mass by looking at general solutions to the quantum mechanical equations in the classical as well as the relativistic domain.

THE SCHRÖDINGER EQUATION

a) The Classical Equation

The idea of standing matter-waves (the foundation of wave mechanics) was developed by Erwin Schrödinger in 1926. The Schrödinger equation describes the wave mechanics that play the same roll in the quantum-world as Newton's laws play in the macro-world. Like Newton's laws, Schrödinger's equation was not derived – it started as a postulate. Jammer [1] describes, in great detail, the progress of the reasoning and development of the equation.

In his first paper [18], Schrödinger modified a classical Hamilton-Jacobi equation. He applied it to the hydrogen atom, producing a discrete eigenvalue spectrum that was equal to the Bohr energy spectrum.

In his second paper [19], Schrödinger developed what he called undulatory Mechanics. He showed that a direct extension of ordinary mechanics into undulatory optics failed for the infinitely small wavelengths, but, assuming an exponential time dependence for his wave ($\exp[2\pi i(E/\mathrm{h})t]$), he produced an equation consistent with his earlier work. Applying his theory to the linear harmonic oscillator, he obtained results in full agreement with Heisenberg's matrix mechanics. He recognized that having the wave-packet phase velocity equal the group velocity was an important relationship that could be used to establish a more intimate connection between wave propagation and the motion of a representative point (the point-like particle).

In his third paper [20], Schrödinger developed his time-independent perturbation theory which could be considered an extension of Rayleigh's method of acoustic vibrations. He applied his technique to the Stark effect of the hydrogen atom, obtaining results agreeing with observations; and, in his fourth paper [21], realizing that total energy, E, varies when passing from one stationary state to another, Schrödinger expressed the wave function as a product of independent temporal and spatial functions producing the familiar time-dependent Schrödinger equation.

The quantum mechanical laws of motion are embodied in the Schrödinger equation [22]:

$$\mathrm{H}\Psi = [-(\hbar^2/2m)\nabla^2 + V(\boldsymbol{r})]\Psi = i\hbar\,\partial\Psi/\partial t = E\Psi. \tag{4}$$

H is the Hamiltonian operator, $\nabla^2 = \partial^2/\partial^2 x + \partial^2/\partial^2 y + \partial^2/\partial^2 z$, and $\hbar = h/2\pi$. The wave function, $\Psi(\boldsymbol{r},t)$, with single frequency and wavelength, is the quantity analogous to displacement for particles. The wave packet acts as a particle, where the wave function is interpreted as the amplitude of the particle-wave, with $\Psi^2(\boldsymbol{r}, t)$ as its maximum, and $\boldsymbol{r}$ as its most probable position. The total energy for the packet is $E = \boldsymbol{p}^2/2m$, and the packet speed is $\boldsymbol{v} = \boldsymbol{p}/m$.

For (4) to have a physical interpretation, several things are necessary:

- Ψ must be single valued in $\mathbf{r}$ and finite everywhere;
- Both Ψ and its derivative must be continuous everywhere;
- $|\Psi|^2$ must be interpreted as a probability density;
- Ψ should be square-integrable ($\int|\Psi|^2$ dr = 1). That is, Ψ is said to be normalized, and the quantity $|\Psi|^2$ is interpreted directly as the spatial probability density of the particle.

Under these conditions this equation is valid for both free particles and particles acted on by a conservative force.

Full details for the solutions to the well-known classical problems described by (4) can be found in standard quantum mechanics texts [22-24]. One example is the one-dimensional potential barrier where (4) takes the form

$$\text{(5)} \qquad d^2U/dx^2 = (2m/\hbar^2)[V - E]\,U = k^2\,U$$

when the wave function $\Psi(x, t)$ is treated as separable in space and time, that is,

$$\Psi(x, t) = U(x)\exp(-i\,E\,t/\hbar).$$

General solutions satisfying (5) are

$$\text{(6)} \qquad U_A(x) = A\exp(k_A x) \qquad \text{or} \qquad U_B(x) = B\exp(-k_B x).$$

Linear differential equations say that if U_A and U_B are separately solutions, then any combination of $U_A \pm U_B$ is also a solution. The details of the full solution to a particular problem depend on the specific boundary conditions. However, there are four general possibilities that immediately come to mind when identifying $k = \sqrt{[(2m/\hbar^2)(V - E)]}$.

They include:

1. For $V > E$ with $m > 0$, then k is real.
2. For $V < E$ with $m > 0$, then k is imaginary.

3. For $V > E$ with $m < 0$, then k is imaginary.
4. For $V < E$ with $m < 0$, then k is real.

This gives two classes of solutions – one when k is real, and one when k is imaginary.

When k is real, the composite solutions for (6) take the form (for A = B)

$$U_+ = U_A + U_B \Rightarrow A' \cosh(kx)$$

and

$$U_- = U_A - U_B \Rightarrow A' \sinh(kx).$$

In both cases, U_+ and U_- satisfy the requirements for being finite, but only within a limited region (such as the finite rectangular potential barrier [22]).

When k is imaginary, we get $k = ik' = i\sqrt{[(2m/\hbar^2)\,|E - V|]}$. For the U_+ case,

$$U_+ = U_A + U_B \Rightarrow A' \cos(k'x).$$

Note that because the cosine function is cyclic, $A' \sin(k'x)$ is also a solution. For the U_- case,

$$U_- = U_A - U_B \Rightarrow A'\, i \sin(k'x)$$

and similarly $A'\, i \cos(k'x)$ is also a solution. The imaginary solution is traditionally viewed as a rotation in phase space, but generally not considered.

Peacock [25,Chapt. 6.1] notes that general plane waves form a complete set so the wave function $\mathbf{\Psi}$ (for (4)) must be a complex quantity because the differential operator is a mixture of real spatial derivatives and an imaginary time derivative. Further, when he first obtained the time-independent equation for which $\mathbf{\Psi}$ can be real, Schrödinger reluctantly considered the possibility of complex solutions for his time-dependent case. Note also that the form of Schrödinger's equation does not preclude the possibility of solutions with negative mass, but negative mass solutions have generally been disregarded as nonphysical – though the Dirac model does consider it, and it will be discussed later.

In general, the potential used in many problems is real (as in the $1/\boldsymbol{r}$ Coulomb potential). However, there are cases where complex potentials are useful. These are found primarily in 3D scattering problems (as in the case of neutron scattering [24, Chapt. 5.20]). When considering the scattering of two particles where the temporal and spatial parts of the wave function are separable, two equations of motion result – one describing the motion of a particle representing the center of mass of the system, and one describing the motion of a free particle that acts like one with the combined mass of the two scattering particles.

When the potential is entirely real, simulations show behavior that looks like the interaction of two soliton-like particles – the closer the masses of the two particles, the more soliton-like the behavior of the scattering. However, when the potential is complex ($V = V_R - i\,V_I$), the situation is quite different. Within certain asymptotic regions, the incoming and the scattered waves interfere with a phase angle leaving a shadow. This shadow is related to the rate at which one particle is absorbed by the other (a neutron being absorbed by one nucleus to produce a different nucleus). The complex potential is referred to as an Optical-Model potential and is analogous to the use of a complex refractive index employed in partially absorbing optical media.

b) The Relativistic Equation

At the time Schrödinger proposed his classical wave equation, he also proposed an extension to it that was consistent with special relativity. This extension was consistent with the laws of mechanics. The total energy for a free particle [24, Chapt. 13.51] becomes:

(7) $$E^2 = c^2\boldsymbol{p}^2 + m^2c^4$$

where the rest mass energy mc^2 is included. Making the usual substitutions ($E \to i\hbar\, \partial/\partial t$, and $\boldsymbol{p} \to i\hbar\nabla$), and setting $V = 0$, the relativistic form of the equation becomes:

(8) $$-\hbar^2\, \partial^2\Psi/\partial^2 t = -\hbar^2c^2\,\nabla^2\Psi + m^2c^4\,\Psi$$

The plane wave solution $\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, satisfies (8) with eigen values $\hbar k$ and $\hbar\omega$ when

(9) $$\hbar\omega = \pm\sqrt{(\hbar^2c^2\boldsymbol{k}^2 + m^2c^4)}.$$

The positive sign on the square root gives the positive energy solution, and the negative sign gives the negative energy solution.

Note that in the non-relativistic case, the position probability density is defined as

$$\mathbf{P}(\boldsymbol{r}, t) = \Psi^*\,\Psi = |\Psi(\boldsymbol{r}, t)|^2\,.$$

But in the relativistic case $\mathbf{P}(\boldsymbol{r}, t)$ becomes

$$\mathbf{P}(\boldsymbol{r}, t) = i\hbar/2mc^2\,[\Psi^*\,\partial\Psi/\partial t - \Psi\,\partial\Psi^*/\partial t].$$

Since this quantity is not necessarily positive, it cannot be interpreted as a probability density; but, if multiplied by the electric charge, *e,* it becomes the electric charge density.

THE KLEIN-GORDON EQUATION

The Klein–Gordon equation (sometimes referred to as the Klein–Gordon–Fock) is a relativistic wave equation, related to the Schrödinger equation [26]. For a free particle its form is:

(10a) $$\hbar^2 c^2 \nabla^2 \Psi - m^2 c^4 \Psi = \hbar^2 \partial^2 \Psi / \partial t^2$$

or alternately expressed as

(10b) $$(\Box^2 + m^2 c^2 / \hbar^2)\, \Psi = 0.$$

The $\Box^2$ is the D'Alembertian operator (squared) [2, Chapt. 6.3] where

$$\Box^2 = \Sigma_\mu\, \partial^2/\partial x_\mu^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - 1/c\, \partial^2/\partial t^2$$

where Σ_μ indicated summation over μ.

The equation was initially proposed by Schrödinger in 1926. (It was also proposed, independently, by the Swedish physicist O. Klein, the Soviet physicist V. A. Fok, and the German physicist W. Gordon.) Schrödinger tried to generalize deBroglie's waves to handle bound particles, obtaining the fine line structure of the hydrogen atom, but the answers failed to fit reality.

Jammer notes [1, p257] that the results were not correct because he did not take into account the spin of the electron (mainly because electron spin was not known at that time). The idea of spin was first introduced by Pauli even though it was not at first clear why the magnetic moment of the electron had to be taken as $\hbar e/2mc$ where e is the charge on an electron.

The non-relativistic Schrödinger equation is based on the energy relation $E = \boldsymbol{p}^2/2m$ where the spatial momentum is $\boldsymbol{p} = m\boldsymbol{v}$. The corresponding relativistic relation is $E^2 = (c\,\boldsymbol{p})^2 - (m\,c^2)^2$. This is the starting point for the Klein-Gordon equation which is actually a four-vector form of the Schrödinger equation (which also does not include spin – so neither equation properly describes the spin-1/2 fermions). The Hamiltonian is the same for both, $H = H_s = \boldsymbol{p}^2/2m$, except that, for the relativistic case, $\boldsymbol{p} \rightarrow \hbar / i\, \nabla$ and $E \rightarrow i\, \hbar\, \partial t$, and the covariant form for the equation becomes

(10c) $$p^\mu\, p_\mu\, \Psi = (mc)^2\, \Psi$$

This has a solution [27, Relativistic Wave Equation]

(11) $$\psi = \mathrm{A} \exp(-i\, p_\mu\, x_\mu)$$

which results in an energy expression,

(12) $$E = \pm(m^2 + \boldsymbol{p} \cdot \boldsymbol{p})^{1/2}.$$

The m^2 allows either positive or negative mass without affecting the energy value – though negative mass is generally not considered. The apparent impossibility of negative values for E was problematic because there was no intuitive interpretation for the meaning of these negative energy states. Further, Peacock notes [25, Chapt. 6.2] because the differential equation is of second order in time, its Ψ-time evolution is completely different in nature from the Schrödinger equation – that is, it was not uniquely determined by its initial values. This problem was avoided by letting the wave function have two components, $\psi_{\pm} = \psi \pm i/m\, \partial\psi/\partial t$, with a 4-current, defined as $J^{\mu} = \psi^{*}\partial^{\mu}\psi - \psi\,\partial^{\mu}\psi^{*}$. This gives the density as

$$\rho_{KG} = J_0 = \psi^{*}\,\partial\psi/\partial t - \psi\,\partial\psi^{*}/\partial t$$

which can be negative, so it cannot be intepreted as a probability density as it is in non-relativistic quantum mechanics where it is defined $\rho_S = \Psi_S^{\dagger}\,\Psi_S$ (which is positive definite). As a result, Ψ was, at first, erroneously interpreted as the wave function. The solution to the equation, $\psi(x, y, z, t)$, is a function only of the spatial coordinates (x, y, z) and the time (t). Consequently, the particles described by this function have no other internal degrees of freedom—that is, they are spinless and cannot describe spin-1/2 fermions.

Because of these problems, the equation was rejected as basis for a (possible) relativistic quantum mechanics. This led Dirac to develop a new relativistic equation which did provide a satisfactory interpretation that could also be applied to the Klein-Gordon equation. The difficulties are overcome by assuming Ψ describes more than a single particle, and it was resurrected in Quantum Field Theory as a valid description of zero-spin composite particles (like the π-meson and the K-meson), and where ρ_{KG} can be both positive and negative because $e \cdot \rho_{KG}$ can be interpreted as an operator for charge density (e = electric charge).

The theory of Quantum Mechanics, in the classical limit, consists of material particles [24, Chapt. 14]. The wave function, for a single particle, is specified by the time dependence of its amplitudes at all points in space – in much the same way as classical particles is specified by their positional coordinates **r**(x, y, z) and their dependence on time. In Quantum Field Theory, the field has an infinite number of degrees on freedom and is analogous to a system containing an infinite number of particles. Field quantization techniques can be applied to the ψ of the wave mechanics equation, converting it from a one-particle system to a many-particle system. However the new field theory formalism deals with processes that involve creation or destruction of material particles (radioactive beta decay, meson-nuclear interactions) which the earlier quantized version of wave mechanics does not. The formalism and relevance to negative mass solutions for Quantum Field Theory [24, Chapt. 14], [25, Chapt. 7], [28] is not included at this time.

THE DIRAC EQUATION

In 1928, Dirac[29] presented an operator approach for the relativistic quantum mechanics. His investigation was centered on resolving the discrepancy between the point-charge electron model and experimental observations. He acknowledged the point-charge model satisfied the requirements of both relativity and the general transformation theory (a technique for connecting the matrix representation of quantum mechanics with the solutions to Schrödinger's wave equation [24, Chapt. 6.23], [27, Algebra of the γ Matrix]. But, he noted the spinning electron model, at least to a first approximation has merit. He further noted the (Klein-)Gordon operator for the wave equation was satisfactory so far as emission and absorption of radiation are concerned, but is not so general as the interpretation of the non-relativity quantum mechanics.

In his paper, he went on to develop theory for the electron. Dirac required that the relativity theory be just as general as the non-relativity theory, and that it define a wave function that was linear in time (as the non-relativity wave function was). For the free electron case, Dirac's approach starts with

(13) $$\mathrm{H}\,\Psi = i\hbar\,\partial\Psi/\partial t$$

[24, Chapt. 13.52]. If the classical relativistic Hamiltonian of a free particle is

(14) $$E = \pm(c^2\boldsymbol{p}^2 + m^2c^4)^{1/2}$$

and $\boldsymbol{p} \rightarrow i\hbar\nabla$, the wave equation is not symmetric with respect to space and time derivatives, making it not relativistic. So Dirac modified the Hamiltonian to be

(15) $$\mathrm{H} = c\,(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2,$$

transforming the equation for a free particle to

(16a) $$(E - c\,(\boldsymbol{\alpha}\cdot\boldsymbol{p}) - \beta mc^2)\,\Psi = 0$$

or alternatively,

(16b) $$(i\hbar\,\partial/\partial t + i\hbar\,c\,(\boldsymbol{\alpha}\cdot\nabla) - \beta mc^2)\,\Psi = 0$$

In (16a), the $\boldsymbol{\alpha}$ and β are matrices of arbitrary numbers that are determined by solving the equation

$$\{(E + c\,(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2)\} \times \{(E - c\,(\boldsymbol{\alpha}\cdot\boldsymbol{p}) - \beta mc^2)\,\Psi = 0\}.$$

He required that $a_i a_j = \beta^2 = 1$ for i and j = 1, 2, 3, and that $\boldsymbol{\alpha}$ and β anti-commute – that is,

$a_i\beta + \beta a_i = 0$ for $i = 1, 2, 3$, and $a_i a_j + a_j a_i = 0$ for $i \neq j$. Solving for the matrices gives

(17)
$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \alpha_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
$$\alpha_y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \qquad \alpha_z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

The waveform is a four-element column matrix

$$\Psi(\mathbf{r},t) = \begin{bmatrix} \Psi_1(\mathbf{r},t) \\ \Psi_2(\mathbf{r},t) \\ \Psi_3(\mathbf{r},t) \\ \Psi_4(\mathbf{r},t) \end{bmatrix}$$

with the adjoint defined as a four-element row matrix containing the complex conjugates

$$\Psi(\mathbf{r}, t)^\dagger = [\ \Psi_1(\mathbf{r}, t)^* \quad \Psi_2(\mathbf{r}, t)^* \quad -\Psi_3(\mathbf{r}, t)^* \quad -\Psi_4(\mathbf{r}, t)^*\ ]$$

The plane wave solution, with j = 1, 2, 3, 4, is

(18) $$\Psi_j(\boldsymbol{r}, t) = S\, u_j \exp[-i/\hbar\,(\boldsymbol{p}\cdot\boldsymbol{r} - E\,t)].$$

S is a normalizing factor that depends on E and m where both E and m can be positive or negative. $\Psi(\mathbf{r}, t)$ has two sets of solutions for each energy sign.

For the positive energy, $E_+ = +(c^2\boldsymbol{p}^2 + m^2c^4)^{1/2}$, $S = 1/(E_+ + mc^2)$ for all u_j's:

(19 a, b)

Spin-up set	Spin-down set
$u_1 = 1$	$u_1 = 0$
$u_2 = 0$	$u_2 = 1$
$u_3 = cp_z$	$u_3 = c(p_x - i\,p_y)$
$u_4 = c(p_x + i\,p_y)$	$u_4 = -\,cp_z$

For the negative energy, $E_- = -(c^2\boldsymbol{p}^2 + m^2c^4)^{1/2}$, $S = 1/(E_- - mc^2)$ for all u_j's:

(20 a, b)

Spin-up set	Spin-down set
$u_1 = cp_z$	$u_1 = c(p_x - i\,p_y)$
$u_2 = c(p_x + i\,p_y)$	$u_2 = -\,cp_z$

u_3 = 1 u_3 = 0
u_4 = 0 u_4 = 1

Both the Schrödinger and Dirac relativistic equations allow solutions when the particle has negative kinetic energy and negative rest mass. These solutions correspond to the negative square root of the classical energy equation (7). However, the negative energy solutions, in the case of Dirac, are not ignored as they are in the classical case since nothing prohibits a charged particle from making a radiation transfer from a positive energy to a negative energy state. Unfortunately, the dual solutions imply the possibility of an electron spontaneously jumping between positive and negative energy states.

Dirac addressed the interpretation of the negative energy solutions with the development of his Hole Theory which includes the concept of a sea of negative energy electrons. Because in Fermi-Dirac statistics [22, Chapt. 17] [24, Chapt. 10], the Pauli Exclusion Principle prevents positive energy electrons from decaying into occupied negative energy states, and the vacuum state is considered the minimum energy state, the electron sea must be fully occupied. This requires the Hole Theory to be a many particle ($N = \infty$) system rather than a single particle theory. Solomon [30, 31] considers the effect of perturbing fields on the requirement of a minimum energy vacuum.

Schiff [24, Chapt. 13.53] notes that, assuming no electromagnetic or gravity effect is exerted, a deviation from norm can result when emptying one or more electrons from the negative energy sea. In that case, the negative energy state (otherwise unseen) becomes observable for a negative electron with negative mass and kinetic energy – this led to the Hole Theory of positrons (electrons with positive charge). He notes Pauli and Weisskopf showed that quantizing the field energy is always positive even if E in the wave equation is positive or negative. So Schrodinger's relativistic equation can also predict particles with positive E that can have both positive and negative charge.

Feynman [32] formalized this by reinterpreting Dirac's negative-energy solutions as representing the electrons moving backward in time giving them a reversed (positive) electric charge. His goal was to keep only positive energies but maintain the positron interpretation. To do this, he applied a transformation $\boldsymbol{p} \rightarrow -\boldsymbol{p}$ and $E \rightarrow -E$. This had two effects on the negative-energy solutions. The normalization factor S changed from $1/(E_- - mc^2) \rightarrow 1/(E_+ + mc^2)$, and, the exponential in (18) changed from $\exp[-i/\hbar(\boldsymbol{p}\cdot\boldsymbol{r} - Et)] \;\rightarrow\; \exp[i/\hbar(\boldsymbol{p}\cdot\boldsymbol{r} - Et)]$, while the u_js in (20 a, b) remained unchanged. The transformed solutions are interpreted as describing antimatter which has positive mass (like the electron) but positive charge (opposite from the electron).

Debergh [33] reconsider the negative-energy solutions providing other transformation possibilities and another interpretation to the results. Their analysis indicates the existence of two types of antimatter – the "classical" one created in the laboratory, and, a "primordial" (negative) antimatter which is composed of negative mass and energy. They argue this negative antimatter is responsible for the acceleration of the cosmic expansion, the confinement of positive mass objects, and the spiral galactic structure, and more.

RELATED MODIFIED SCHRÖDINGER EQUATIONS

Finally, as we wrap up this survey, we briefly discuss two modified versions of the Schrödinger equation, one with a complex coefficient that demonstrates its relation to the diffusion equation, and another nonlinear version with soliton solutions.

a. A Schrödinger Type Equation with Complex Coefficients

Schrödinger began his work trying to generalize de Broglie's waves to the case of bound particles. He recognized the structure of his wave equation as a diffusion equation with an imaginary coefficient. He considered the possibility of a complex coefficient, but selected the simplest equation that a wave would obey noting that it was possible to interpret a particle as a wave packet so long as one could neglect any spreading [19]. Okino [34] shows how it is possible to derive Schrödinger's equation from the diffusion equation by making the substitution $t \rightarrow i\,t$, that is time becomes imaginary. Fracsa [35] show the existence of a diffusion process described by a Schrödinger equation. This equation was reduced from a Chapman-Kolmogorov equation with a complex diffusion coefficient. Their modified Brownian motion simulation technique has the potential to provide insight into the behavior of negative-mass component particles embedded in a positive-mass cloud (of the type described in the mixed-mass composite elementary particles in a dual universe [11]).

Here, we consider a modification of Schrödinger's equation, replacing the usual coefficient with a complex one for the one-dimensional case.

(21) $$(\mathrm{A} + i\,\mathrm{B})\,[\nabla^2 \;+\; V(x)]\,\Psi \;=\; \partial\Psi/\partial t.$$

Writing the general solution as $\Psi(x, t) = \mathrm{U}(x)\,\mathrm{T}(t)$ gives

(22) $(1/U)\,(A + iB)\,[\nabla^2 U + VU)]\,\Psi = (1/T)\,\partial\Psi/\partial t = E$ = constant.

The temporal solution has the form $T \propto \exp(Et)$. The spatial solution where V = constant has two possible forms:

(22a) $U \propto \exp[-ex + \delta]$, where $(A + iB)(e^2 + V) = E$

and

(22b) $U \propto \sin[-dx + \delta]$, where $(A + iB)(-d^2 + V) = E$

Consider the behavior of the two solutions where e and d are real and $e > 0$.

If $B = 0$, for the special case of $V(x) = 0$, (21) reduces exactly to the diffusion equation when $A = k^2$ where k is the diffusion coefficient:

(23) $k^2\,\partial\Psi^2/\partial x^2 = \partial\Psi/\partial t$

For real values of k, the solution becomes

(24) $\Psi(x, t) \propto \exp[-(dk)^2 t]\,\sin(dx)$

This solution decays exponentially with time (t) with a sinusoidal oscillation in space (x).

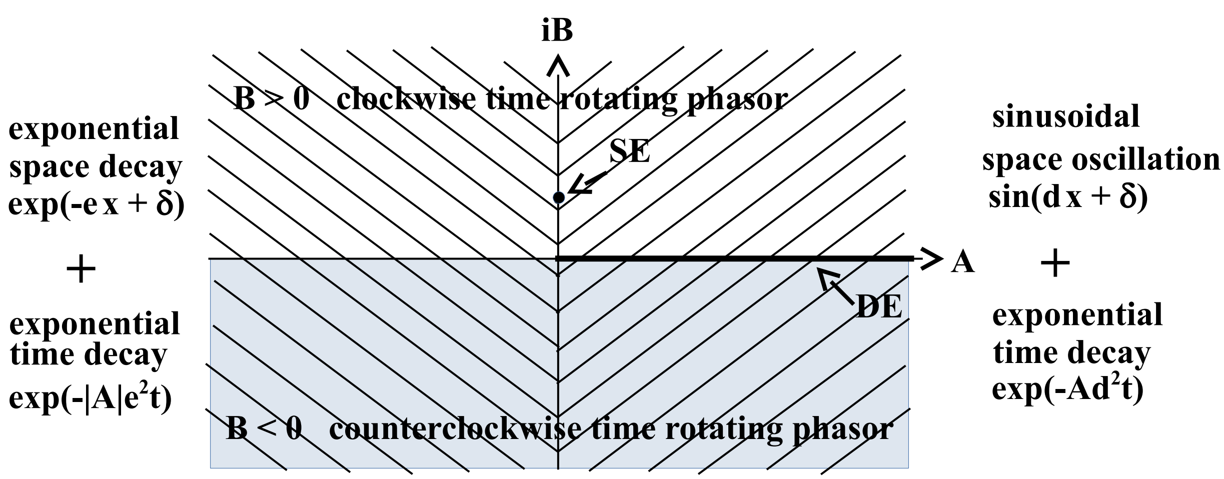
If $A = 0$, (21) reduces exactly to the Schrödinger equation when $B = \hbar/2m$, and the equation is multiplied by $i\hbar$, giving

(25) $-(\hbar^2/2m)\,\partial\Psi^2/\partial x^2 + \tilde{V}(x)\Psi = i\hbar\,\partial\Psi/\partial t$

With $\tilde{V}(x) = -2V(x)/\hbar^2$, the solution (22b) reduces to the form

(26) $\Psi(x, t) \propto \exp[-i\omega t]\,\sin(dx - \delta)$

When $\omega = B(d^2 + V)$, the solution has a sinusoidal oscillation in space (x), and a rotating phase angle in time.



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SE = Schrödinger Equation, DE = Diffusion Equation

Figure 1. This shows the regions of validity for the constants A and B when $V = 0$. DE indicates diffusion results for a real diffusion coefficient. SE indicates the Schrödinger equation result with its imaginary coefficient.

b. Solitons and the Nonlinear Schrödinger Equation

Recall, the linear Schrödinger equation (equ. 4 above)

$$[-(\hbar^2/2m)\nabla^2 + V(\mathbf{r})]\,\Psi(\mathbf{r}, \mathrm{t}) = i\hbar\,\partial\Psi(\mathbf{r}, \mathrm{t})\,/\,\partial t$$

describes the quantum mechanical laws of motion where the wave function is interpreted as a wave packet that acts as a particle. We considered a specific one-dimensional case (equ. 5 above) and the possible impact of negative mass. We now address the presence of nonlinearities in the propagation medium.

There are modifications that describe real world complex nonlinear phenomena – including such things as light propagation in optical fibers [36], multi-nucleon and multi-quark interactions [37]. A one-dimensional generalized version of the linear Schrödinger equation [38] is

(27) $$i\,\Psi_x + \tfrac{1}{2}\,\alpha(x)\,\Psi_{tt} + \beta(x)\,|\Psi|^2\,\Psi = i\,\gamma(x)\,\Psi$$

where Ψ is the wave amplitude and the subscripts refer to differentiation with respect to the propagation distance coordinate x, and the time coordinate t. The $\alpha(x)$ is the group velocity dispersion, $\beta(x)$ is the nonlinearity, and $\gamma(x)$ is the gain. For appropriate values of the spatially dependent coefficients, (27) reduces to the well known nonlinear Schrödinger equation

(28) $$i\,\Psi_x + \tfrac{1}{2}\,\mathrm{D}\,\Psi_{tt} + |\Psi|^2\,\Psi = 0$$

This classical field equation is known to model, to first order, propagation in nonlinear materials (such as an optical fiber), and is known to have soliton solutions. The $\mathrm{D} = +1$ indicates anomalous dispersion, while $\mathrm{D} = -1$ indicates normal dispersion. There are two well known stationary solutions to (28):

(29a) $$|\Psi|^2 = A^2\,\mathrm{sech}^2(\tau) \text{ when } \mathrm{D} = +1,$$

and

(29b) $$|\Psi|^2 = A^2\,\tanh^2(\tau) \text{ when } \mathrm{D} = -1$$

where τ is the retarded time $\tau = t - \beta z$. The z is axial distance down the fiber, t is time, β is the transmission speed of the pulse, and A^2 is the maximum field intensity.

It is well known that narrow pulses experience temporal broadening resulting from the dispersive properties of the medium in which they propagate. In a nonlinear medium, the nonlinear properties can cause pulse compression that cancels the broadening. That allows it to maintain its shape while propagating at a constant velocity. This is a soliton or solitary wave – that is, a self-reinforcing wave packet. It should be noted that solitons maintain their shape even when they interact with another soliton by forward collision or by being overtaken from behind [39].

Bright soliton propagation was first verified by Mollenauer *et al.* [40] over a short distance in an effectively lossless optical fiber over a short. The formation of dark solitons in the normal dispersion regime was verified by Krokel *et al.* [41] for a narrow dark pulse superimposed on a wide Gaussian-shaped background pulse. Their results showed the dark portion of the waveform propagated as a soliton while the Gaussian background spread.

In the case of propagation in an optical fiber, the non-linearity is the result of the second order susceptibility of the induced polarization of the electric field. This allows the index of refraction for the fiber with a Kerr nonlinearity to be written as $n = n_0 + n_2|E|^2$. This is the source of the third term in (28) for the fiber case – where $n_2|E|^2$ is not explicitly dependent on time and is small compared to n_0.

The stationary solutions (29a, b) result when the wave amplitude is purely real. Giannini and Joseph [42] show a transformation of a complex amplitude that leads to a family of exact non-stationary solutions to (28) that take the form of Painleve transcendents of the second kind [43]. These solutions are both bounded and stable in shape for the anomalous group velocity dispersion (bright solitons), but, for the normal group velocity dispersion (dark solitons), bounded solutions exist only over a finite range of the nonlinearity to dispersion.

For the case when the amplitude has an imaginary part, this can contribute to loss in the medium (causing spreading and attenuation) and (28) can be written

(30) $$i\Psi_x + \tfrac{1}{2}D\Psi_{\tau\tau} + |\Psi|^2\Psi = -i\Gamma\Psi$$

Giannini and Joseph [42] developed an analytic solution for this using perturbation techniques to find the additional terms to (29a) and (29b) resulting from the loss (Γ). Their solutions show spreading and attenuation in the dark pulse is slower than for the bright pulse. For example, with a parameter value Γ = ~1.5, the Fig. 2 shows the spreading for a bright pulse, and Fig. 3 shows the corresponding spreading for the dark pulse. In both figures, the initial pulse (indicated as pulse width = 1.0, x = 0) would be a soliton pulse propagating in a zero loss medium (where Γ = 0). In these figures, the amplitudes ($|\Psi|^2$), propagation distances (x), and retarded time (τ) are scaled values. Further, the complex amplitude causes greater absolute spreading during propagation than an equivalent pulse with a purely real amplitude.

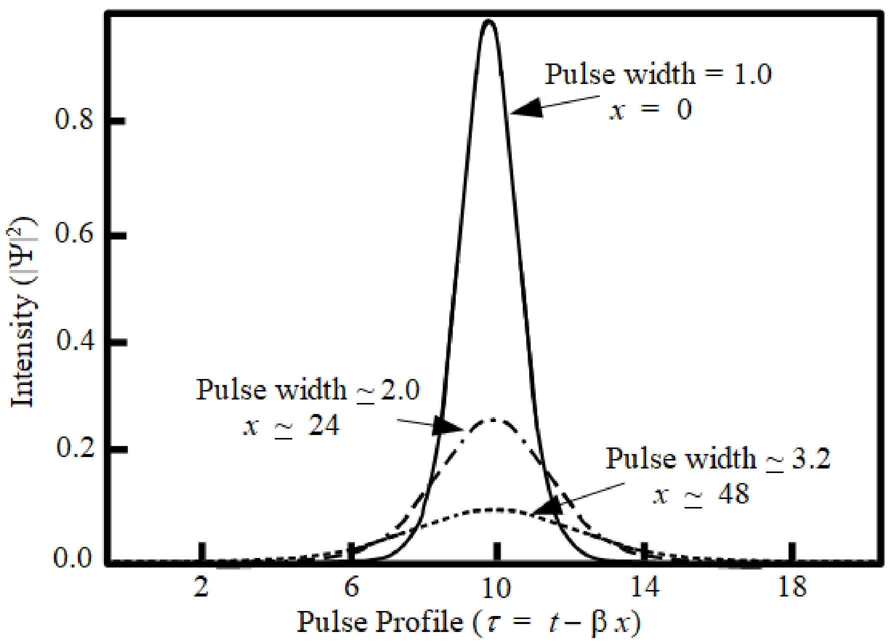

Figure 2. This shows an example of the spreading in an initial bright pulse at three distances in the propagation as a result of loss in the fiber. At x = 24 (scaled units), the initial pulse has spread to twice its initial width.

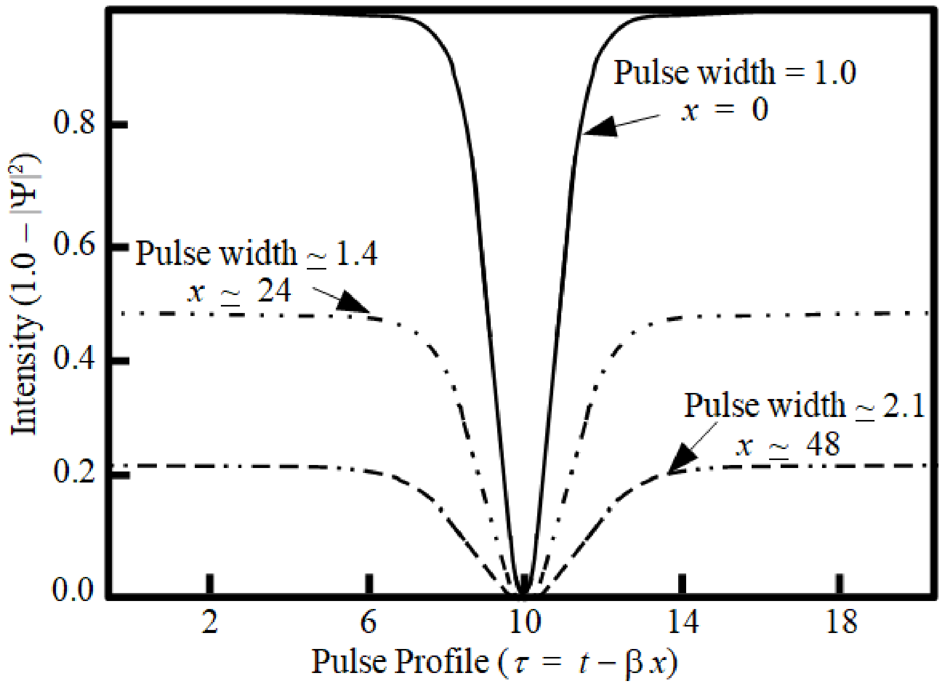

Figure 3. This shows an example of the spreading in an initial dark pulse at three distances in the propagation as a result of loss in the fiber. At x = 48 (scaled units), the initial pulse has spread to twice its initial width – spreading twice as slowly as the corresponding bright pulse.

Finally, Yaroslav,et al. [45] survey the progress of soliton research in nonlinear lattices (NLs), which represent spatially periodic patterns of modulation in the local strength and sign of the nonlinearity. A large part of the work is focused on nonlinear optics (light waves) and Bose-Einstein condensates (BECs) like ultracold vapors of alkali metals (matter waves). In BEC, one-dimensional and multidimensional NLs may be induced by the application of spatially periodic external fields that induce a corresponding modulation pattern of the local nonlinearity.

In optical media, Nls may be built as material structures, represented by spatially periodic modulations of the local Kerr coefficient. For BECs, the description is of bosonic atom interactions in rarefied degenerate gases ("degenerate" means that the de Broglie wavelength of atoms is comparable to the mean interatomic distance). A generalized equation for this effect is:

(31) $$[-(\hbar^2/2m)\nabla^2 + U(\mathbf{r}, t)]\,\Psi + (\beta(\mathbf{r})/m)\,|\Psi|^2\,\Psi = i\hbar\,\partial\Psi/\partial t.$$

where m is the atomic mass, $U(\mathbf{r}, t)$ is the external potential acting on individual atoms, and β is the scattering length which determines collisions between the atoms. For $\beta > 0$ and $\beta < 0$, the interactions are repulsive and attractive, respectively. Unlike the nonlinear Schrödinger equation used for light waves (4), the corresponding equation for matter waves (31) explicitly shows the mass dependence which can be positive or negative.

In the mixed-mass composite elementary particles of the Dual Universe model (proposed in [11]), the repulsion and collision between the positive mass and negative mass components would cause the composite elementary particles to be unstable. This would result in the breakup of the larger particles (matter waves) into smaller matter wave unites (the decay particle products). Some of these smaller particles would be stables, others would not. The fractal structure of the component parts of the composite particles might be represent as a nonlinear lattice, and, theory analogous to the BEC theories summarized by Yaroslav might be applicable in describing the negative mass – positive mass interaction of the composite model. This is the subject of on-going research.

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