Physics Letters A, Vol. 141, Issue 8-9, 417-419, DOI: 10.1016/0375-9601(89)90860-8, 1987

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Abstract: A transformation of the complex amplitude of the propagating electric fields in an optical fiber shows a family of exact non-stationary solutions to the nonlinear Schrodinger equation. These solutions take the form of the second Painleve transcendent. In the case of normal group velocity dispersion (dark solitons), bounded solutions exist only over a finite range of the ratio of the nonlinearity to dispersion.

The propagation of optical pulses in single mode lossless dispersive fibers is described by the nonlinear Schrodinger equation:

$$U_z = \mathrm{i}(\pm U_{tt} - 2U|U|^2. \qquad (1)$$

The $U(z, t)$ is the complex amplitude of the electric field and the subscripts on U indicate partial derivatives with respect to the indicated variables. The positive sign on the U_{tt} term corresponds to normal group velocity dispersion and leads to a dark pulse. The negative sign corresponds to anomalous dispersion, representing a bright pulse. The recent development of techniques for shaping short pulses [1-3] has generated renewed interest in the theory of dark solitons and their development has been studied numerically [4].

This Letter presents a transformation of the nonlinear Schrodinger equation resulting in a family of analytic non-stationary solutions for pulse propagation which are related to the second Painleve transcendent. The solutions are applicable to both normal and anomalous dispersion but interest here is in the normal dispersion case (dark solitons).

The amplitude U of the electric field, modeled by the nonlinear Schrodinger equation, can be written in the form:

$$U(z, t) = R(z, t) \exp[\mathrm{i}\theta(z, t)], \qquad (2)$$

where $R(z, t)$ is the amplitude of the complex number and the $\theta(z, t)$ is the phase. For the stationary case leading to the well known soliton solutions [5], transformation to a reference frame fixed with respect to the moving wave uses the coordinate

$$\zeta = t - z/v,$$

where v is the speed of the waves. The non-stationary solutions derived here are obtained by the using the transformation

$$\eta = t + z(f + gz) \quad (3)$$

This transforms the solution to a reference frame that is fixed with respect to an accelerating wave. The form for the phase θ(z, t) is chosen to be

$$\theta = az + bt + czt + dz^2 + ez^3. \quad (4)$$

where a, b, c, d, e, f and g are free parameters. Combining eqs. (1)–(4) and choosing parameters such that

$$b = \mp f/2, \quad c = \mp g, \quad d = \mp fg, \quad e = \mp 2g^2/3$$

results in the following equation for R:

$$\pm R'' = R(a \pm f^2/4 \mp g\eta) + 2R^3, \quad (5)$$

where the prime denotes differentiation with respect to η. Note that if $g = 0$, eqs. (3) and (5) yield the usual stationary solutions to the nonlinear Schrodinger equation. The final transformation of eq. (5) uses

$$x = (\eta - q)/p \quad \text{and} \quad y(x) = R(\eta)/A, \quad (6)$$

where

$$q = (f^2/4 \pm a)/g, \quad p^3 = -1/g, \quad A^2 = g^2/3.$$

This gives a family of solutions for y (and thus for $R(\eta)$) in the form of the Painleve transcendents of the second type,

$$y'' = xy \pm 2y^3, \quad (7)$$

where the prime denotes differentiation with respect to x. Note that in this form, the positive sign on the cubic term corresponds to normal (positive) group velocity dispersion.

This characterization of the solutions to the nonlinear Schrodinger equation, although addressed in the hydrodynamic context, has not been associated previously with optical pulse propagation. Smith [6], who modeled large amplitude deep water waves, examined the behavior of the general asymptotic character of the solutions to an equation similar to eq. (1),

$$\mathrm{i}a_t = a_{\rho\rho} - \rho a + \beta|a|^2 a.$$

Although his results were obtained for a steady state, it is easy to show that by writing his amplitude as

$a = B \exp(\mathrm{i}\theta)$,

and using the transformation

$\eta = \rho + t^2$ and $\theta = t(\rho + t^2/3)$,

his equation can be written in the form

$\mathrm{i}B_t = B_{\eta\eta} + \beta|B|^2B$,

and the solutions in terms of Painleve transcendent can be obtained without the steady state restriction. Ablowitz [7] recognized this same behavior in his analysis of solutions to the Korteweq–de Vries equation, and, Miles [8] and Rosales [9] examined the Painleve equation (7) noting that in the asymptotic regime

$y(x) = \alpha\,\mathrm{Ai}(x) \quad (x \to \infty)$,

where Ai(x) is the Airy function of the first kind. This provides the initial conditions for y and y', for the numerical integration of eq.(7) using the corresponding values of the Airy function at a sufficiently large x (x = 10). (For the case of normal group velocity dispersion, the function y corresponds to what Miles and Rosales call F_{-}). These solutions are bounded over all x when $0 < \alpha < 1$ for the case of interest here (dark solitons), decaying monotonically to zero in the positive x-direction. In the negative direction, the solutions oscillate as they decay. As $\alpha \to 0$, y/α approaches the Airy function for all x. As $\alpha \to 1^{-}$, y/α becomes unbounded as $x \to -\infty$. Just as for the case of the Korteweg–de Vries equation [8], the parameter α is a measure of the relative importance of the nonlinearity and the dispersion in the system. Fig. 1 shows the square of the integrated values of y/α (proportional to the intensity) for several values of α (α = 0.1, 0.9 and 0.999).

In conclusion, the transformation presented here leads to a family of non-stationary solutions to the nonlinear Schrodinger equation that are both bounded and stable in shape.

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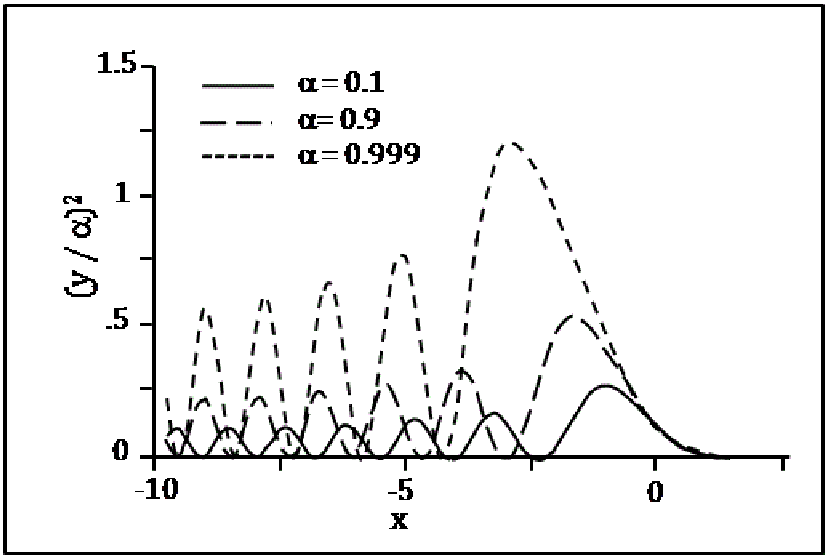

Fig. 1 Integrated values of *y/a* for three values of *a*. As *a* –> 0, *y/a* approaches the Airy function. As *a* –> 1, *y/a* becomes inbounded at *x* = –00.